MEDIAN-SHIFT SIGN STATISTICS FOR CONSTANT SIGNAL DETECTION IN IMPULSIVE NOISE ENVIRONMENT

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Abstract

In this paper, we propose a new detector of which the basis is on the median-shift sign. We call it the median-shift sign (MSS) detector, which is a generalization of the classical (standard) sign detector. First, we consider the asymptotic optimum median shift (MS) values and their characteristics. Next we consider the asymptotic relative efficiency (ARE) of the MSS detectors.

1 Introduction

The problem of signal detection can be considered as a parameter testing problem of a null hypothesis against an alternative hypothesis [1]. As a consequence, the knowledge of *a priori* information on the parameter is required for establishing the hypothesis testing problem. Unfortunately, it is very difficult, if not impossible, to exactly estimate the value of a parameter in practice. If we are not able to get a priori information on the distribution of the parameter, we cannot design an optimum parametric detector. Although we can estimate the parameters in some cases, small deviations of the parameters from the theoretic model in the real environment may lead sometimes to a significant performance degradation of the optimum parametric detector. In such cases, we shall need a *nonparametric* detector [2]-[7].

In this paper, we propose and analyze the asymptotic characteristics of a nonparametric detector acquired by modifying the sign detector.

2 The Observation Model

Consider the binary hypothesis testing problem: given an observation vector $\underline{X}_n = (X_1, X_2, ..., X_n)$, a decision is to be made between a null hypothesis H and an alternative hypothesis K. The pair H and K are defined as

$$\begin{aligned} H: X_i &= N_i, & i = 1, 2, \cdots, n, \\ K: X_i &= N_i + S, & i = 1, 2, \cdots, n. \end{aligned} \tag{1}$$

In (1), n is the sample size, $S > 0$ is a constant representing the signal, and N_i are the independent and identically distributed random variables representing noise components, each with zero-mean probability density function (pdf) f.

3 The Median-Shift Sign Test Statistic

Now we consider the test statistic of the detector proposed in this paper,

$$T_{MSS}(\underline{X}_n) = \sum_{i=1}^{n} U(X_i + V), \tag{2}$$

where

$$U(z) = \begin{cases} 1 & , \text{if } z > 0, \\ 0 & , \text{if } z < 0, \end{cases} \tag{3}$$

and V is the *median-shift (MS) value.* We call the detector based on the test statistic (2) the *median-shift sign (MSS)* detector. The *optimum MS value* V_{op} is so obtained as to make the detection probability maximum, once the sample size n, false alarm rate α, signal

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strength S, and noise pdf f are fixed. When $V = V_{op}$, the MSS detector will be called the *optimum MSS* detector. We will denote the MSS detector with the MS value V by the MSS (V) detector: note that the MSS (0) detector is the sign detector.

The probability that the input data plus MS value V is positive under the alternative hypothesis is given by

$$\begin{aligned} P_1 &= \int_0^\infty f\{x - (S+V)\}dx \\ &= F(S+V), \end{aligned} \tag{4}$$

where F is the cumulative distribution function (cdf) of noise. Under the null hypothesis, $S = 0$ and we have

$$\begin{aligned} P_0 &= P_1|_{S=0} \\ &= F(V). \end{aligned} \tag{5}$$

We can now obtain the threshold and randomization parameter using (4) and (5). The threshold λ is the smallest integer which satisfies

$$\sum_{k=\lambda+1}^{n} \binom{n}{k} P_0^k (1-P_0)^{n-k} \leq \alpha, \tag{6}$$

where $1 \leq \lambda \leq n$ and α is the false alarm rate. The randomization parameter γ is given by

$$\gamma = \frac{\alpha - \sum_{k=\lambda+1}^{n} \binom{n}{k} P_0^k (1-P_0)^{n-k}}{\binom{n}{\lambda} P_0^\lambda (1-P_0)^{n-\lambda}}, \tag{7}$$

where $0 \leq \gamma < 1$.

Then, using λ and γ, we can compute the detection probability as follows:

$$\begin{aligned} P_D &= \sum_{k=\lambda+1}^{n} \binom{n}{k} P_1^k (1-P_1)^{n-k} \\ &\quad + \gamma \binom{n}{\lambda} P_1^\lambda (1-P_1)^{n-\lambda}. \end{aligned} \tag{8}$$

From (8), the optimum MS value is obtained by

$$V_{op} = arg \max_V P_D, \tag{9}$$

which is a function of the sample size n, false alarm rate α, signal strength S, and the noise pdf f.

4 Asymptotic optimum MS value

Let us now consider some properties of the asymptotic optimum MS value. Consider the generalized Gaussian (GG), generalized Cauchy (GC), and generalized logistic (GS) distributions, whose pdf's are given by

$$\begin{aligned} f_{GG}(x) &= \frac{k}{2A_{GG}(k)\Gamma(1/k)} e^{-[\frac{|x|}{A_{GG}(k)}]^k}, \quad (10) \\ f_{GC}(x) &= \frac{B(k,\nu)}{\{1 + [|x|/A_{GC}(k)]^k/\nu\}^{\nu+1/k}}, \quad (11) \end{aligned}$$

and

$$f_{GS}(x) = \frac{\Gamma(2k)}{\sigma_{GS}[\Gamma(k)]^2} \frac{e^{-\frac{kx}{\sigma_{GS}}}}{(1+e^{-\frac{x}{\sigma_{GS}}})^{2k}}, \tag{12}$$

respectively, where

$$\begin{aligned} A_{GG}(k) &= [\sigma_{GG}{}^2 \frac{\Gamma(1/k)}{\Gamma(3/k)}]^{1/2}, \quad (13) \\ \sigma_{GG} &= k\sqrt{\frac{\pi}{2}} \frac{[\Gamma(3/k)]^{1/2}}{[\Gamma(1/k)]^{3/2}} \sigma, \quad (14) \\ A_{GC}(k) &= [\sigma_{GC}{}^2 \frac{\Gamma(1/k)}{\Gamma(3/k)}]^{1/2}, \quad (15) \\ B(k,\nu) &= \frac{k\nu^{-1/k}\Gamma(\nu+1/k)}{2A_{GC}(k)\Gamma(\nu)\Gamma(1/k)}, \quad (16) \\ \sigma_{GC} &= k\sqrt{\frac{\pi}{2}} \frac{\nu^{-\frac{1}{k}}\Gamma(\nu+1/k)[\Gamma(3/k)]^{1/2}\sigma}{\Gamma(\nu)[\Gamma(1/k)]^{3/2}}, \quad (17) \end{aligned}$$

and

$$\sigma_{GS} = \frac{\sqrt{2\pi}}{4^k} \frac{\Gamma(2k)}{[\Gamma(k)]^2} \sigma. \tag{18}$$

Here $k > 0$, $\nu > 0$, Γ is the Gamma function, and σ is the *common deviation parameter* introduced to make $f_{GG}(0) = f_{GC}(0) = f_{GS}(0) = f(0)$. Depending on the values of k, σ, and ν, the pdf's (10)-(12) represent a wide spectrum of light- and heavy-tailed pdf's, which are useful in the modeling of impulsive noise environment. They also include some commonly-used well-known pdf's as special cases. For example, the generalized Gaussian pdf becomes the Gaussian pdf with $\sigma_G = \sigma_{GG}|_{k=2} = \sigma$ and the Laplace pdf with $\sigma_L = \sigma_{GG}|_{k=1} = \sqrt{\pi}\sigma$ when $k = 2$ and $k = 1$, respectively. The generalized Cauchy pdf becomes the Cauchy pdf with $\sigma_C = \sigma_{GC}|_{k=2,\nu=0.5} = \sqrt{\frac{2}{\pi}}\sigma$ when $k = 2$ and $\nu = 0.5$. Similarly, the generalized logistic pdf becomes the logistic pdf with $\sigma_S = \sigma_{GS}|_{k=1} = \sqrt{\frac{\pi}{8}}\sigma$ when $k = 1$.

Now we consider the case where the sample size n becomes very (infinitely) large. It is well-known that the distributions of $(T_{MSS} - E\{T_{MSS}\})/\sqrt{Var\{T_{MSS}\}}$ converge, as the sample size tends to infinity, to the standard Gaussian distribution $N(0,1)$ by DeMoivre-Laplace theorem or the central limit theorem. Then we can compute the detection probability in the large sample-size case. We have

$$\begin{aligned} \alpha &= Pr(T_{MSS} > \lambda | H) \\ &\approx 1 - \Phi\left(\frac{\lambda - nP_0}{\sqrt{nP_0(1-P_0)}}\right), \end{aligned} \quad (19)$$

from which we get

$$\lambda \approx nP_0 + \sqrt{nP_0(1-P_0)}\Phi^{-1}(1-\alpha), \quad (20)$$

where Φ is the standard normal cdf. Therefore, we have

$$\begin{aligned} P_D &= Pr(T_{MSS} > \lambda | K) \\ &\approx \Phi\left(-\frac{\sqrt{n}(P_0 - P_1) + \sqrt{P_0(1-P_0)}\Phi^{-1}(1-\alpha)}{\sqrt{P_1(1-P_1)}}\right). \end{aligned} \quad (21)$$

Some comparisons between (8) and (21) are shown in Figure 1: it should be noted that the approximations are very close to the exact ones.

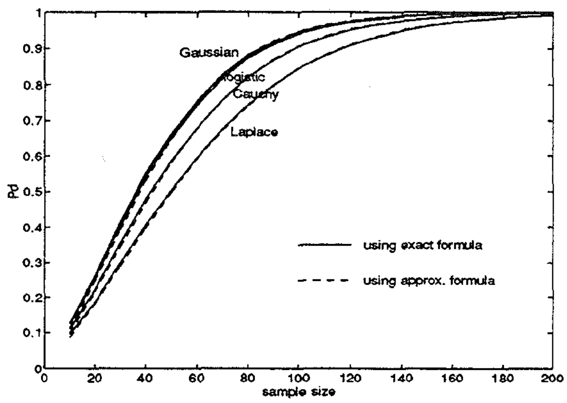

Figure 1: Comparisons between P_D using exact formula and P_D using approx. formula

Now, defining

$$D(S,V) = \frac{\{F(S+V) - F(V)\}^2}{F(S+V)\{1 - F(S+V)\}}, \quad (22)$$

it is straightforward to show that

$$\begin{aligned} arg \max_V P_D &= arg \max_V \frac{(P_1 - P_0)^2}{P_1(1-P_1)} \\ &= arg \max_V D(S,V) \end{aligned} \quad (23)$$

as $n \to \infty$. Since $lim_{V \to \pm\infty} D(S,V) = 0$ and $D(S,V)$ is not a constant, there exists a maximum. It is clear to see by differentiation that a necessary condition for $V = V_{as}$ is

$$\begin{aligned} &2F(S+V)\{f(S+V) - f(V)\}\{1 - F(S+V)\} = \\ &\quad f(S+V)\{F(S+V) - F(V)\}\{1 - 2F(S+V)\}, \end{aligned} \quad (24)$$

where $V_{as} = lim_{n\to\infty} V_{op}$. We can show [8] that $V_{as} \geq -\frac{S}{2}$ for cdf's satisfying $F(-x) = 1 - F(x)$ when $S > 0$.

Let us now consider the values of V_{as} for $S \to 0$ and $S \to \infty$: let us denote by V_I and V_F the value of V_{as} for $S \to 0$ and $S \to \infty$, respectively. It is noteworthy that the results for $S \to 0$ should be interesting in locally optimum (LO) detection problems. First, from (24) after some manipulations, we get

$$\frac{2f(V_F)}{1 - F(V_F)} = g_{LO}(\infty), \quad (25)$$

where $g_{LO}(x) = -\frac{f'(x)}{f(x)}$ is the LO nonlinearity of the noise pdf f. When $g_{LO}(\infty)$ is finite, there exists a unique V_F satisfying (25) for unimodal pdf's, since $\lim_{x\to-\infty} \frac{f(x)}{1-F(x)} = 0$, $\lim_{x\to\infty} \frac{f(x)}{1-F(x)} = \lim_{x\to\infty} g_{LO}(x)$ using L'Hospital's rule, and $\frac{f(x)}{1-F(x)}$ is monotone for $x < 0$ and for $x > 0$. When $g_{LO}(\infty)$ is infinite, on the other hand, we have $V_F \to \infty$, since $\lim_{x\to\infty} \frac{f(x)}{1-F(x)} = \lim_{x\to\infty} -\frac{f'(x)}{f(x)} = \lim_{x\to\infty} g_{LO}(x)$.

Next, let us obtain V_I. For $S \to 0$, we have

$$2f'(V_I)F(V_I)\{1 - F(V_I)\} = f^2(V_I)\{1 - 2F(V_I)\} \quad (26)$$

from (24) after some manipulations. Alternatively, we can obtain V_I by maximizing $\frac{\partial^m D(S,V)}{\partial S^m}|_{S=0}$, where m is the first non-zero derivative. It turned out that $m = 2$ and we are thus to maximize $\frac{\partial^2 D(S,V)}{\partial S^2}|_{S=0} = \frac{2f^2(V)}{F(V)\{1-F(V)\}}$: that is, we have

$$V_I = arg \max_V \frac{f^2(V)}{F(V)\{1 - F(V)\}}. \quad (27)$$

Clearly, for many (but not all) noise pdf's f satisfying $f'(0) = 0$ and $F(0) = 0.5$, we have $V_I = 0$ from (26), which is true for some typical unimodal symmetric pdf's including the Gaussian, Laplace, and Cauchy pdf. For some other unimodal symmetric pdf's and multimodal pdf's, however, $V_I \neq 0$ in general. For example, consider the pdf's

$$f_1(x) = \frac{1}{2}\{f_G(x-1) + f_G(x+1)\}, \quad (28)$$

$$f_2(x) = \frac{1}{2}|x|e^{-|x|}, \quad (29)$$

$$f_3(x) = \begin{cases} f_3(-x) & \text{, for } x < 0, \\ -\frac{5}{6264}x^4 + \frac{35}{232} & \text{, for } 0 \le x < 3, \\ \frac{5}{58}e^{-(x-3)} & \text{, for } 3 \le x, \end{cases} \quad (30)$$

$$f_4(x) = \begin{cases} f_4(-x) & \text{, for } x < 0, \\ -\frac{1}{6}x^3 + \frac{5}{12} & \text{, for } 0 \le x < 1, \\ \frac{1}{4}e^{-2(x-1)} & \text{, for } 1 \le x, \end{cases} \quad (31)$$

$$f_5(x) = \begin{cases} f_5(-x) & \text{, for } x < 0, \\ -0.02x + 0.265 & \text{, for } 0 \le x < 1, \\ 0.245e^{-(x-1)} & \text{, for } 1 \le x, \end{cases} \quad (32)$$

and

$$f_6(x) = \begin{cases} -\frac{|x|}{2} + \frac{3}{4} & \text{, for } |x| \le 1, \\ 0 & \text{, for } |x| > 1, \end{cases} \quad (33)$$

where $f_G(x)$ is the standard normal pdf. All the pdf's f_i, $i = 1, 2, \cdots, 6$ are symmetric, f_1 and f_2 are bimodal, f_3 and f_4 are unimodal and differentiable, f_5 is unimodal but not always differentiable, and f_6 is unimodal and has finite non-zero support. Then we obtain $V_I \approx 1.1601$, 1.1483, 2.352, 0.6201, 1, and 1, respectively,

The values of V_I and V_F for various pdf's are shown in Table 1 for easy reference. Figure 2 shows the signal strength versus the values of V_{as} when $\sigma = 3.0$. Since f_6 has a finite non-zero support, we do not include the results in this figure. As clearly shown in this figure, V_I is non-zero for $f_i(x), i = 1, 2, \cdots, 5$, although they satisfy $f'(0) = 0$ and $F(0) = 0.5$: we would again like to emphasize that $V_I = 0$ for only some of the unimodal symmetric pdf's.

5 Asymptotic Performance Comparisons

Now we consider the ARE of the MSS detectors. The detection probabilities of the MSS (V_1) and MSS (V_2) detectors are, when n_1 and n_2 are sufficiently large,

$$P_{D1} \approx \Phi\left[\frac{\sqrt{F(V_1)\{1-F(V_1)\}}\Phi^{-1}(1-\alpha) - \sqrt{n_1}\{F(S+V_1) - F(V_1)\}}{\sqrt{F(S+V_1)\{1-F(S+V_1)\}}}\right]$$

and

$$P_{D2} \approx \Phi\left[\frac{\sqrt{F(V_2)\{1-F(V_2)\}}\Phi^{-1}(1-\alpha) - \sqrt{n_2}\{F(S+V_2) - F(V_2)\}}{\sqrt{F(S+V_2)\{1-F(S+V_2)\}}}\right],$$

Table 1: Some values of V_I and V_F

		V_I	V_F
GG	$k > 1$	0	∞
	$k = 1$	0	$\sqrt{\frac{\pi}{2}}\sigma \ln \frac{2}{3}$
	$k < 1$	0	$-\infty$
GC		0	$-\infty$
GS		0	0
$f_1(x)$		1.1601	∞
$f_2(x)$		1.1483	1
$f_3(x)$		2.352	1.3548*
$f_4(x)$		0.6201	0.2035*
$f_5(x)$		1	$\frac{\sqrt{3673}-61}{4} \approx 0.09868$
$f_6(x)$		1	*undefined*

*These numbers are the solutions in the interval [0,1] to $x^5 + 10x^4 - 945x + 1242 = 0$ and $x^4 + 4x^3 - 10x + 2 = 0$, respectively.

respectively. Therefore,

$$ARE_{1,2} = \lim_{n_1 \to \infty, n_2 \to \infty, S \to 0} \frac{n_2}{n_1}|_{P_{D1} = P_{D2}} = \frac{E(V_1)}{E(V_2)}, \quad (34)$$

where

$$E(V) = \frac{f^2(V)}{F(V)\{1 - F(V)\}}. \quad (35)$$

Note that $E(0) = 4f^2(0)$ is the efficacy of the sign detector for symmetric pdf's: thus (35) can be regarded as a simple expression for the efficacy of the MSS detectors. As we can see from (27), (34), and (35), $E(V)$ is maximum at $V = V_I$, and therefore we have $ARE_{opt,sign} \ge 1$ for the ARE of the optimum MSS detector with respect to the sign detector: the equality holds when $V_I = 0$ as for some of the well-known unimodal symmetric pdf's.

Some values of the AREs are shown in Figure 3. Since $E(0) = 0$ and consequently $ARE_{MSS,sign} \to \infty$ for $V \ne 0$ if the noise pdf is f_2, we do not provide the results for f_2 in this figure. Apparently, if the noise pdf is such that $V_I = 0$, then $ARE_{MSS,sign} \le 1$. If, on the other hand, the noise pdf is such that $V_I \ne 0$, then $ARE_{MSS,sign}$ may have a value greater than 1. In other words, it is noteworthy that, depending on the pdf, even the AREs of some non-optimum MSS detectors with respect to the sign detector are greater than 1: for example, if the pdf is bimodal as shown in (28), then any MSS (V) detector with V approximately in the interval (0, 1.7),

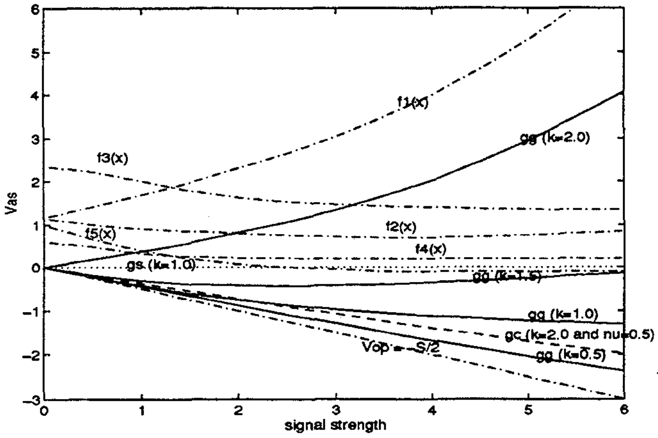

Figure 2: Signal strength versus optimum value when n is infinity and $\sigma = 3.0$

which may or may not be the optimum MSS detector, is asymptotically better than the sign detector, as shown in Figure 3. Similarly, for the unimodal symmetric pdf f_4, any MSS (V) detector with V approximately in the interval (0, 0.8) would perform asymptotically better than the sign detector.

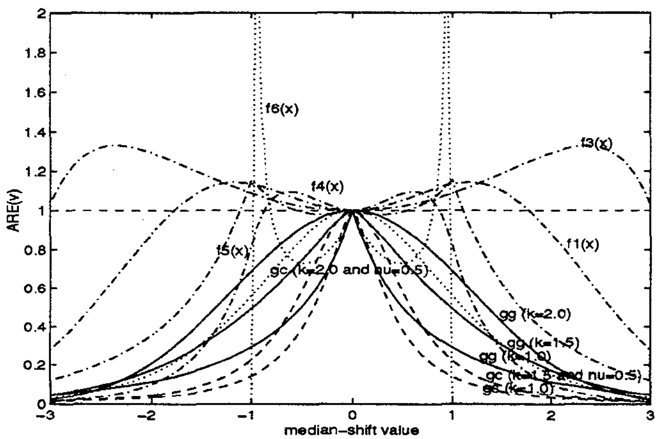

Figure 3: Median-shift value versus ARE when noise is bimodal as shown in (28)

6 Conclusion

In this paper, we proposed a new detector based on the MS sign statistic. This detector was a modification and an extension of the sign detector. We showed some properties of the asymptotic optimum MS value. We then considered the ARE of the MSS detectors. It is noteworthy that, depending on the pdf, even the AREs of some non-optimum MSS detectors with respect to the sign detector are greater than 1.

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