Generalized Crosscorrelation Properties of Chu Sequences

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***Abstract*—In this paper, we present a theorem that shows generalized properties for the cross-correlation function of Chu sequences. The theorem gives the information on the maximum magnitude and its position of cross-correlation function for all kinds of Chu sequences with any length. By using this theorem, we design the pilot sequences for multi-cell environment. It mitigate effectively interference and we can confirm the superiority of the proposed scheme to the conventional scheme.**

***Index Terms*—Chu sequences, Maximum magnitude of cross-correlation function, Pilot sequence, Multi-cell, Interference.**

I. Introduction

Generally, in order to permit unambiguous message synchronization, to minimize cochannel interference, and to support a large number of active user, large families of sequences with good auto-correlation function and small cross-correlation function values are required. A periodic sequences' auto-correlation function which is 0 except for the period-multiple-shift terms is called perfect auto-correlation function [1]. Chu sequences [2] are class of sequences with perfect auto-correlation. Furthermore, a lower bound of maximum magnitude of the cross-correlation function is obtained in [3] and it is proved that Chu sequences satisfy the lower bound in specific cases [4]. However, we cannot always select Chu sequences satisfying the lower bound, because the number of Chu sequences which satisfy the lower bound is limited. In section II, we present a theorem representing more generalized cross-correlation properties of Chu sequences. The theorem gives the information on the cross-correlation function of all kinds of Chu sequences with any length. It is expected that these properties give a useful guideline for the efficient mitigation of co-channel interference. In section III, we proposed the channel estimation technique using the theorem for multi-cell environment. It mitigate effectively multi-cell interference due to the characteristic of the pilot sequence's correlation.

II. Correlation properties of Chu sequences

A set of Chu sequences with length N is defined as $C = \{a_r\}$ where $a_r = \{a_r(0), a_r(1), ..., a_r(N-1)\}$ and

$$a_r(k) = \begin{cases} \exp\left(i\pi \dfrac{rk^2}{N}\right) & , \ N\ even, \\ \exp\left(i\pi \dfrac{rk(k+1)}{N}\right), & N\ odd. \end{cases} \tag{1}$$

Here, $0 \le k, r < N$ and r is relatively prime with N.

A. Periodic autocorrelation function

The autocorrelation function with lag j is defined as,

$$x_r(j) = \sum_{k=0}^{N-j-1} a_r(k)a_r^*(k+j) + \sum_{k=N-j}^{N-j-1} a_r(k)a_r^*(k+j-N), \tag{2}$$

where it was shown that Chu sequences have the following properties [3].

$$x(j) = \begin{cases} N, & j \mod N = 0 \\ 0, & j \mod N \neq 0 \end{cases} \tag{3}$$

B. Periodic cross-correlation function

Let a_r and a_s be any Chu sequences. Then, the cross-correlation function $y_{r,s}(j)$ of a_r and a_s with lag j is defined as

$$y_{r,s}(j) = \sum_{k=0}^{N-j-1} a_r(k)a_s^*(k+j) + \sum_{k=N-j}^{N-j-1} a_r(k)a_s^*(k+j-N). \tag{4}$$

When N is even, we can rewrite (4),

$$\begin{aligned} y_{r,s}(j) = &\sum_{k=0}^{N-j-1} \exp\left(i\pi\frac{rk^2}{N}\right) \exp\left\{-i\pi\frac{s(k+j)^2}{N}\right\} \\ &+ \sum_{k=N-j}^{N-1} \exp\left(i\pi\frac{rk^2}{N}\right) \exp\left\{-i\pi\frac{s(k+j-N)^2}{N}\right\}. \end{aligned} \tag{5}$$

It is proved in [2] that for an arbitrary integer d,

$$\exp\left\{i\pi\frac{r(k+d+N)^2}{N}\right\} = \exp\left\{i\pi\frac{r(k+d)^2}{N}\right\}. \tag{6}$$

is satisfied. Thus, (5) can be rewritten as

$$y_{r,s}(j) = \sum_{k=0}^{N-1} \exp\left(i\pi\frac{rk^2}{N}\right) \exp\left\{-i\pi\frac{s(k+j)^2}{N}\right\}, \tag{7}$$

The following lemmas will be useful for proof of theorem 1.

***Lemma** 1:* A primitive hth root of unity ξ is defined as

$$\xi = \exp\left(i2\pi\frac{u}{h}\right), \tag{8}$$

where u is any integer relatively prime to h. It can be easily for any integer v, $0 < v \le h-1$, the following relation is valid [3].

$$\sum_{k=0}^{h-1} \zeta^{\pm vk} = 0, \quad \zeta \neq 1. \tag{9}$$

***Lemma** 2:* If $l = k + e$, $e = 0, 1, \cdots, N$ and a function $f(x)$ is periodic with period N, then the follow equation is satisfied.

$$\sum_{l=0}^{N-1} f(l) = \sum_{e=0}^{N-1} f(e). \tag{10}$$

Substituting $k + e$ in $\sum_{l=0}^{N-1} f(l)$, we obtain

$$\sum_{l=0}^{N-1} f(l) = \sum_{e=k}^{N+k-1} f(e). \tag{11}$$

Since $f(x)$ is periodic with period N, (10) is rewritten as

$$\sum_{l=0}^{N-1} f(l) = \sum_{e=k}^{N-1} f(e) + \sum_{e=0}^{k-1} f(e) = \sum_{e=0}^{N-1} f(e). \tag{12}$$

In this way, we proved the Lemma 2.

Then, the following theorem holds.

***Theorem** 1:* Let $G = \gcd(N, r-s)$, $N = c_1 G$, $r - s = c_2 G$ and $j = c_3 G + d$, where c_1 is prime with c_2 and $0 \le c_3 \le c_1$. Then, the squared absolute value of the cross-correlation function $y_{r,s}(j)$, $|y_{r,s}(j)|^2$, is given as

$$|y_{r,s}(j)|^2 = \begin{cases} NG, & if\, N\, even,\, c_1 c_2\, even,\, d = 0 \\ NG, & if\, N\, even,\, c_1 c_2\, odd,\, d = \dfrac{G}{2} \\ NG, & if\, N\, odd,\, d = 0 \\ 0, & otherwise \end{cases}$$

Proof: We can rewrite (7) as

$$\begin{aligned} y_{r,s}(j) =& \exp\left(-i\pi \frac{sj^2}{c_1 G}\right) \\ & \cdot \sum_{k=0}^{c_1 G-1} \exp\left\{ i2\pi \left(\frac{c_2 k^2}{2c_1} - \frac{skc_3}{c_1} - \frac{skd}{c_1 G} \right) \right\}. \end{aligned} \tag{13}$$

Since $|y_{r,s}(j)|^2 = y_{r,s}(j) y_{r,s}(j)^*$,

$$\begin{aligned} |y_{r,s}(j)|^2 =& \sum_{k=0}^{c_1 G-1} \sum_{l=0}^{c_1 G-1} \exp\left(i2\pi \left\{ \frac{c_2 k^2}{2c_1} - \frac{sk(c_3 G + d)}{c_1 G} \right\} \right) \\ & \cdot \exp\left(i2\pi \left\{ \frac{sl(c_3 G + d)}{c_1 G} - \frac{c_2 l^2}{2c_1} \right\} \right). \end{aligned} \tag{14}$$

We can know that the last term of (14) is periodic with period $c_1 G$ from follows.

$$\begin{aligned} & \exp\left(i2\pi \left\{ \frac{s(l + c_1 G)(c_3 G + d)}{c_1 G} - \frac{c_2 (l + c_1 G)^2}{2c_1} \right\} \right) \\ =& \exp\left(i2\pi \left\{ \frac{sl(c_3 G + d)}{c_1 G} - \frac{c_2 l^2}{2c_1} \right\} \right) \\ & \exp\left(-i2\pi \left\{ \frac{c_2 c_1 G^2}{2} \right\} \right) \exp\left(i2\pi (c_3 G + d + l\, G c_2) \right) \\ =& \exp\left(i2\pi \left\{ \frac{sl(c_3 G + d)}{c_1 G} - \frac{c_2 l^2}{2c_1} \right\} \right). \end{aligned} \tag{15}$$

If N and $r - s$ are even, $G = \gcd(r - s, N)$ is always even. Therefore, the last equality of (15) is valid. Consequently, we obtain $|y_{r,s}(j)|^2$ from Lemma 2 and (15) as follows and divide in two parts for convenience.

$$\begin{aligned} \left|y_{r.s}(j)^2\right| =& \sum_{e=0}^{c_1 G-1} \exp\left(i2\pi \left\{ -\frac{c_2 e^2}{2c_1} + \frac{se(c_3 G + d)}{c_1 G} \right\} \right) \\ & \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi \frac{c_2 ke}{c_1} \right) \\ =& A_{r,s}(j) + B_{r,s}(j) \\ A_{r,s}(j) \triangleq& \sum_{e=mc_1} \exp\left(i2\pi \left\{ -\frac{c_2 e^2}{2c_1} + \frac{se(c_3 G + d)}{c_1 G} \right\} \right) \\ & \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi \frac{c_2 ke}{c_1} \right) \\ B_{r,s}(j) \triangleq& \sum_{e \neq mc_1} \exp\left(i2\pi \left\{ -\frac{c_2 e^2}{2c_1} + \frac{se(c_3 G + d)}{c_1 G} \right\} \right) \\ & \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi \frac{c_2 ke}{c_1} \right). \end{aligned} \tag{16}$$

Substituting $e = mc_1$ in $A_{r,s}(j)$,

$$\begin{aligned} A_{r,s}(j) =& \sum_{m=0}^{G-1} \exp\left\{ i2\pi \left(-\frac{c_1 c_2 m^2}{2} + smG + \frac{smd}{G} \right) \right\} \\ & \cdot \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi c_2 km \right) \\ =& c_1 G \sum_{m=0}^{G-1} \exp\left(-\frac{c_1 c_2 m^2}{2} + i2\pi \frac{smd}{G} \right). \end{aligned} \tag{17}$$

The last equality is derived, since $\sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi c_2 km\right) = c_1 G$. Also, we assume that $c_1 c_2$ is even, then

$$A_{r,s}(j) = c_1 G \sum_{m=0}^{G-1} \exp\left(i2\pi \frac{smd}{G} \right). \tag{18}$$

If $d = 0$, $A_{r,s}(j = c_3 G) = c_1 G^2$. But, when $d \neq 0$, since s is relatively with G, we can know easily that $A_{r,s}(j \neq c_3 G) = 0$

from Lemma 1. For $e \neq mc_1$, $B_{r,s}(j)$ is satisfied from Lemma 1 that

$$G \sum_{k=0}^{c_1-1} \exp\left(-i2\pi \frac{c_2 ke}{c_1}\right) = 0. \tag{19}$$

Therefore, we always obtain that $B_{r,s}(j) = 0$ regardless of j. Then, we can derive by (16) that

$$|y_{r,s}(j = c_3 G)|^2 = A_{r,s}(j = c_3 G) = c_1 G^2, \tag{20}$$

$$|y_{r,s}(j \neq c_3 G)|^2 = 0. \tag{21}$$

Now, we consider the case that $c_1 c_2$ is odd in (17). For this case, if $d = \frac{G}{2}$, then

$$\begin{aligned} A_{r,s}(j = c_3 G + \frac{G}{2}) &= c_1 G \sum_{m=0}^{G-1} \exp\left\{ i2\pi\left(-\frac{c_1 c_2 m^2 + sm}{2}\right)\right\} \\ &= c_1 G \sum_{m=0}^{G-1} \exp\left\{ i2\pi\left(-\frac{m(c_1 c_2 m + s)}{2}\right)\right\}. \end{aligned} \tag{22}$$

In (22), s and $c_1 c_2$ is odd. If m is odd, $c_1 c_2 m$ is also odd and $c_1 c_2 m + s$ is even since an odd number added to an odd number make an even number. Therefore, $m(c_1 c_2 m + s)$ is even. Conversely, if m is even, $m(c_1 c_2 m + s)$ is even,too. In other word, $m(c_1 c_2 m + s)$ is always even and then (22) is rewritten as

$$A_{r,s}(j = c_3 G + \frac{G}{2}) = c_1 G^2. \tag{23}$$

If $d \neq \frac{G}{2}$, we obtain that

$$\begin{aligned} &A_{r,s}(j \neq c_3 G + \frac{G}{2}) \\ &= c_1 G \sum_{m=0}^{G-1} \exp\left\{ i2\pi \left(-\frac{m(c_1 c_2 m + s)}{2} + \frac{smc_4}{G}\right)\right\}, \end{aligned} \tag{24}$$

where $0 < c_4 < G$. Previously, since we know that $m(c_1 c_2 m + s)$ is even, (24) is expressed as

$$A_{r,s}(j \neq c_3 G + \frac{G}{2}) = c_1 G \sum_{m=0}^{G-1} \exp\left(i2\pi \frac{smc_4}{G}\right). \tag{25}$$

From Lemma 1 and (25), we obtain that

$$A_{r,s}(j \neq c_3 G + \frac{G}{2}) = 0. \tag{26}$$

When $c_1 c_2$ is odd, $B_{r,s}(j \neq c_3 G + \frac{G}{2})$ is same with the case that $c_1 c_2$ is even. Therefore, by (16)

$$\left| y_{r,s}\left(j = c_3 G + \frac{G}{2}\right)\right|^2 = A_{r,s}(j \neq c_3 G + \frac{G}{2}) = c_1 G^2, \tag{27}$$

$$\left| y_{r,s}(j \neq c_3 G + \frac{G}{2})\right|^2 = 0. \tag{28}$$

On that way we have proved the Theorem 1 when N is even. We can prove the case that N is odd in similar manner and will prove this in appendix A. ■

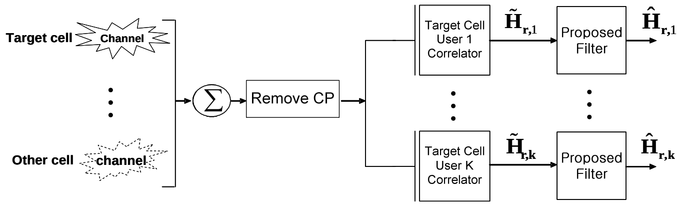

Fig. 1. Proposed Channel Estimation Process

III. Application to pilot patterns for Multi-cell Environemt

In this section, we design the pilot sequences using Chu sequences and receiver structure for multi-cell environment. By (3), we can know that Chu sequences have zero circular autocorrelation function. Therefore, when we transmit a_r added CP, we can estimate the channel impulse response without multi-path interference. Also, if we use Chu sequences in pilots without multiuser interference, the kth user's pilot sequence is kCP cyclic shifted version of a_r. However, if the active user increase or it is considered the user of the other cell like the case that $K > (N/CP)$ where K is the number of the active user, it suffer severely from the interference. Therefore, we require new pilot pattern that effectively can mitigate the interference. The pilot sequences of the target cell and other cell is defined as $p_r = [p_{r,1},\, p_{1,2}, \cdots,\, p_{r,K}]$, $p_s = [p_{s,1},\, p_{s,2}, \cdots,\, p_{s,K}]$

$$\begin{aligned} p_{r,k} &= [a_r(kCP), \cdots,\ a_r(N),\ \cdots,\ a_r(kCP-1)] \\ p_{s,k} &= [a_s(kCP), \cdots,\ a_s(N),\ \cdots,\ a_s(kCP-1)] \end{aligned} \tag{29}$$

where $p_{r,k}$ is kth user's pilot of target cell and $p_{s,k}$ is kth user's pilot of other cell.

The process is represented the proposed channel estimation technique at the receiver in Fig 1. After remove CP, in order to estimate kth user's channel, we have the process which the received signal correlate with the kth user's pilot sequences from zero cyclic shifted sequence to $CP-1$ cyclic shifted sequence. Then the correlation value of received signal and m cyclic shifted kth user's pilot is regard as $\tilde{h}_{r,k}(m)$ where $\tilde{h}_{r,k}(m)$ is m delayed channel's estimated gain value of the target cell's kth user. In this case, we can know that the pilot sequences's correlation of the target cell and other cell influence the interference by theorem 1. Statistically, if $\gcd(r-s, N)$, the correlation's maximum magnitude of the pilot sequences , is minimized, it is expected that the channel impulse response's difference between the target cell and the other cell is largest. In this way, for each individual user, the J channel gain value with largest power will be selected from the output of the correlator and is expressed as

$$\hat{h}_{r,k}(m) = \left\{ \begin{array}{c} \tilde{h}_{r,k}(m),\ if\ \left\{ \left|\tilde{h}_{r,k}(m)\right|^2 \right\} \subset MAX\_M \\ 0,\, else \end{array} \right. \tag{30}$$

where MAX_M is the set of the taps with maximum power.

Through the technique above, we can mitigate effectively

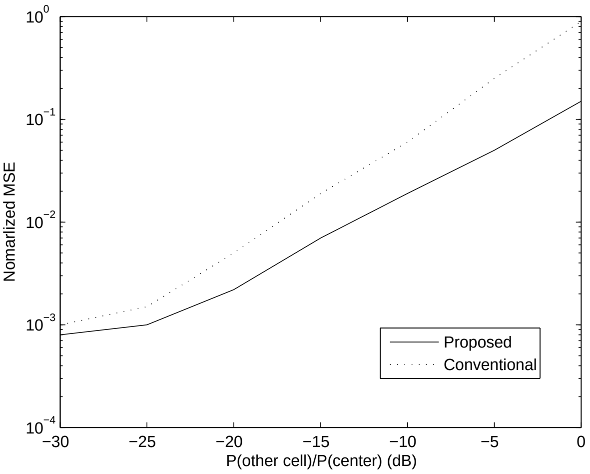

Fig. 2. Comparison of the proposed scheme and the conventional scheme

and get the channel impulse response. To illustrate the performance of the channel estimator, we evaluate the MSE. The MSE performance is evaluated as a function of the average power from adjacent cells. For the simulation, we assume that the length of the pilot sequences is 1024, and pass through Rayleigh fading multi-path channels that are independently generated for base stations. Also, we assume that the number of the adjacent cells is 6 and the length of CP is 128. In order to minimize the maximum power of other cell, we select the pilot sequences as $a_r = a_1$, $a_{s_1} = a_3$, $a_{s_2} = a_7$, $a_{s_3} = a_{11}$, $a_{s_4} = a_{15}$, $a_{s_5} = a_{19}$, $a_{s_6} = a_{23}$ by the proposed scheme in section III. In this case, $\gcd(r - s_m, N) = 2$ regardless of m. For the conventional scheme, we select the pilot sequence of the target cell as $a_r = a_1$ and other cell's pilot sequences randomly of the number that relatively prime with N.

In Fig. 2, the MSE of the proposed scheme and the conventional scheme is compared for signal to other cell interference ratio(SIR). We can confirm that the gap between the proposed and conventional scheme increase with the increase of other cell interference. In other words, we can conclude that the proposed scheme is superior to the conventional scheme in multi-cell environment.

IV. Conclusions

In this paper, we present the generalized cross-correlation properties for Chu sequences. The proved theorem can be applied to all kinds of Chu sequences with any sequence. Also, using this theorem, we propose the pilot pattern for multi-cell environment. Simulation result have confirmed the superiority of the proposed scheme to the conventional scheme.

Appendix A
Proof of the Theorem 1 When N is odd

Consider the case that N is odd. From (1), (4), (6)

$$y_{r,s}(j) = \sum_{k=0}^{N-1} \exp\left(i\pi \frac{rk(k+1)}{N}\right) \cdot \exp\left(-i\pi \frac{s(k+j)(k+1+j)}{N}\right). \tag{31}$$

When N is odd, let $G = \gcd(N, r-s)$, $N = c_1 G$, $r - s = c_2 G$, $j = c_3 G + d$ in similar manner that N is even. Substituting this variables in (31), we obtain

$$y_{r,s}(j) = \exp\left(-i\pi \frac{sj^2 + sj}{c_1 G}\right) \sum_{k=0}^{c_1 G-1} \exp\left(i2\pi \left\{\frac{c_2 k^2}{2c_1} + \frac{c_2 k}{2c_1} - \frac{skc_3}{c_1} - \frac{skd}{c_1 G})\right\}\right). \tag{32}$$

From (32), we can derive $|y_{r,s}(j)|^2$ from Lemma 1 easily in similar manner that N is even. Then we obtain

$$|y_{r,s}(j)|^2 = \sum_{e=0}^{c_1 G-1} \exp\left(i2\pi \left\{-\frac{c_2(e^2+e)}{2c_1} + \frac{se(c_3 G + d)}{c_1 G}\right\}\right) \cdot \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi \frac{c_2 ke}{c_1}\right). \tag{33}$$

In (33), we divide in two parts for convenience like (16) and substitute $e = mc$ in (33),

$$\begin{aligned} A_{r,s}(j) &= \sum_{m=0}^{G-1} \exp\left\{i2\pi \left(-\frac{c_2(m^2 c_1 + m)}{2} + \frac{sm(c_3 G + d)}{G}\right)\right\} \sum_{k=0}^{c_1 G-1} \exp\left(-i2\pi c_2 km\right) \\ &= c_1 G \sum_{m=0}^{G-1} \exp\left\{i2\pi \left(-\frac{c_2 m(c_1+1))}{2} + \frac{sm(c_3 G + d)}{G}\right)\right\}. \end{aligned} \tag{34}$$

An odd number is made up of odd number's multiplication. Therefore, if N is odd, G and c_1 are odd. Then, $c_1 + 1$ is even number and (34) can be rewritten as

$$A_{r,s}(j) = c_1 G \sum_{m=0}^{G-1} i2\pi \frac{smd}{G}. \tag{35}$$

From (35), if $d = 0$, $A_{r,s}(j = c_3 G) = c_1 G^2$. Conversely, if $d \neq 0$ $A_{r,s}(j \neq c_3 G) = 0$ by Lemma 1. Furthermore, we can know that $B_{r,s}(j) = 0$ regardless of d from (33) and Lemma 1. Therefore, we can derive as follows

$$|y_{r,s}(j = c_3 G)|^2 = A_{r,s}(j = c_3 G) = c_1 G^2, \tag{36}$$

$$|y_{r,s}(j \neq c_3 G)|^2 = 0. \tag{37}$$

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