Rank-Based Detection of Random Signals in a Weakly Dependent Noise Model

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Abstract— In this paper, we consider nonparametric signal detection problems under the presence of additive noise exhibiting weak dependence. We derive the test statistic of the locally optimum rank detector under a weakly dependent noise model for random signals. The performance characteristics of the locally optimum rank detector are analyzed in terms of asymptotic relative efficiency.

I. Introduction

Locally optimum (LO) detectors [1]-[5] have been of much interest in signal detection theory and applications because of their capability and effectiveness for detecting weak signals. A subclass of the LO detectors, the locally optimum rank (LOR) detector, has been investigated because an LOR detector requires only simple arithmetic operations, has lower sensitivity to small deviations of the noise pdf, and has nonparametric nature [6]-[8].

It has been commonly assumed that the additive noise samples are statistically independent. In practice, however, this assumption is often violated. Thus, investigations on signal detections in dependent noise are desirable. Among the typical investigations on signal detection problems under various dependent noise models are those in [9]-[12].

In this paper, we will investigate the LOR detector for random signals under a weakly dependent noise model. The weakly dependent noise will be modeled as the first order moving average (MA) of an independent and identically distributed (i.i.d.) random process. The test statistic of the LOR detector will be derived and then the asymptotic performance characteristics of the LOR detector will be investigated.

II. The Observation Model

Let H_0 and H_1 be the null and alternative hypotheses, respectively. Then, the observation model can be written as

$$\begin{aligned} H_0: &\quad X_i = W_i, & i = 1, 2, \cdots, n, \\ H_1: &\quad X_i = \theta s_i + W_i, & i = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where $\{X_i\}$ are the observations, $\{W_i\}$ are the weakly dependent noise components, θ is a signal strength parameter, and $\{s_i\}$ are the signal components. The weakly dependent noise $\{W_i\}$ can generally be modeled by the Volterra expansion [13], which is, however, almost intractable to handle because of the infinitely many terms of the expansion. In this paper, we will assume that the weakly dependent noise W_i, $i = 1, 2, \cdots, n$, are the MA of i.i.d. random variables as

$$W_i = e_i + \rho e_{i-1} u_{i-2}, \tag{2}$$

where e_i, $i = 1, 2, \cdots, n$, are i.i.d. random variables with common pdf f_e. The pdf f_e is even symmetric with bounded continuous derivatives and satisfies the regularity conditions [6]. In (2), ρ is called the dependence parameter determining the correlation coefficient of W_i, and u_i is the unit step sequence defined by $u_i = 1$ when $i \geq 0$ and $u_i = 0$ when $i < 0$.

Let $\underline{X}$, $\underline{W}$, $\underline{e}$, and $\underline{s}$ be the n-tuple vectors representing $(X_1, X_2, \cdots, X_n)$, $(W_1, W_2, \cdots, W_n)$, $(e_1, e_2, \cdots, e_n)$, and $(s_1, s_2, \cdots, s_n)$, respectively, and $f_{\underline{W}}(\underline{W})$, $f_{\underline{e}}(\underline{e}) = \prod_{i=1}^{n} f_e(e_i)$, and $f_{\underline{s}}(\underline{s})$ be the pdfs of $\underline{W}$, $\underline{e}$, and $\underline{s}$, respectively. Then, we have

$$\begin{aligned} f_{\underline{W}}(\underline{W}) &= f_e(W_1) f_e(W_2 - \rho W_1) \cdots \\ &\quad f_e(W_n - \rho W_{n-1} + \cdots + (-\rho)^{n-1} W_1) \\ &= f_e(X_1 - \theta s_1) f_e(X_2 - \theta s_2 - \rho(X_1 - \theta s_1)) \cdots \\ &\quad f_e(X_n - \rho X_{n-1} + \cdots + (-\rho)^{n-1} X_1 \\ &\quad -\theta(s_n - \rho s_{n-1} + \cdots + (-\rho)^{n-1} s_1) \\ &= \prod_{i=1}^{n} f_e(Y_i - \theta c_i) \\ &= f_{\underline{e}}(\underline{Y} - \theta \underline{c}), \end{aligned} \tag{3}$$

where $Y_i = \sum_{k=0}^{i-1} (-\rho)^k X_{i-k}$, $c_i = \sum_{k=0}^{i-1} (-\rho)^k s_{i-k}$, $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$, and $\underline{c} = (c_1, c_2, \cdots c_n)$.

III. The Locally Optimum Rank Detector

Let us define the sign vector $\underline{Z} = (Z_1, Z_2, \cdots, Z_n)$ and the magnitude rank vector $\underline{Q} = (Q_1, Q_2, \cdots, Q_n)$, where $Z_i = sgn(Y_i)$ and Q_i is the rank of $|Y_i|$ in the set $|Y| = \{|Y_1|, |Y_2|, \cdots, |Y_n|\}$. We will also use $|Y|_{[i]}$ to denote the ith smallest member of $|Y|$.

Let the random signal components $\{s_i\}$ form a random process with mean zero and covariance function $r_s(i,j)$. Then, the joint pmf of $(\underline{Q}, \underline{Z})$ is

$$p(\underline{q}, \underline{z}|\theta) = Pr\{\underline{Q} = \underline{q}, \underline{Z} = \underline{z}|\theta\} = \int_B \int_{R^n} f_{\underline{e}}(\underline{Y} - \theta\underline{c}) f_{\underline{s}}(\underline{s}) d\underline{s} d\underline{Y}, \quad (4)$$

where R^n is the set of all n-tuples of real numbers. Then, the LOR test statistic in this random signal case can be calculated as [7]

$$T_{LOR}(\underline{X}) = \left.\frac{d^2 p(\underline{q}, \underline{z}|\theta)}{d\theta^2}\right|_{\theta=0} / p(\underline{q}, \underline{z}|0). \quad (5)$$

From (3) and (4), it is easily seen that

$$\begin{aligned}
\left.\frac{d^2 p(\underline{q}, \underline{z}|\theta)}{d\theta^2}\right|_{\theta=0}
&= \int_B \int_{R^n} \left.\frac{d^2 f_{\underline{e}}(\underline{Y} - \theta\underline{c})}{d\theta^2}\right|_{\theta=0} f_{\underline{s}}(\underline{s}) d\underline{s} d\underline{Y} \\
&= \int_B \int_{R^n} f_{\underline{e}}(\underline{Y}) f_{\underline{s}}(\underline{s}) \left[\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} c_i c_j g_{LO}(Y_i) g_{LO}(Y_j) + \sum_{i=1}^{n} c_i^2 h_{LO}(Y_i)\right] d\underline{s} d\underline{Y} \\
&= \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \int_{R^n} c_i c_j f_{\underline{s}}(\underline{s}) d\underline{s} \int_B z_i z_j g_{LO}(|Y_i|) g_{LO}(|Y_j|) f_{\underline{e}}(\underline{Y}) d\underline{Y} \\
&\quad + \sum_{i=1}^{n} \int_{R^n} c_i^2 f_{\underline{s}}(\underline{s}) d\underline{s} \int_B h_{LO}(|Y_i|) f_{\underline{e}}(\underline{Y}) d\underline{Y} \\
&= \left[\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E_{\underline{s}}\{c_i c_j\} z_i z_j \Gamma_2(Q_i, Q_j) + \sum_{i=1}^{n} E_{\underline{s}}\{c_i^2\} D(Q_i)\right] / (2^n n!) \quad (6)
\end{aligned}$$

and

$$p(\underline{q}, \underline{z}|0) = \int_B \int_{R^n} f_{\underline{e}}(\underline{Y}) f_{\underline{s}}(\underline{s}) d\underline{s} d\underline{Y} = 1/(2^n n!), \quad (7)$$

where $h_{LO}(x) = f_e''(x)/f_e(x)$, $E_{\underline{s}}\{\cdot\}$ is the expectation over $\underline{s}$, $\Gamma_2(i,j) = E\{g_{LO}(|Y|_{[i]}) g_{LO}(|Y|_{[j]})|\theta = 0\}$, and $D(i) = E\{h_{LO}(|Y|_{[i]})|\theta = 0\}$. Thus, the LOR test statistic can be obtained as

$$T_{LOR}(\underline{Y}) = \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} E_{\underline{s}}\{c_i c_j\} z_i z_j \Gamma_2(Q_i, Q_j) + \sum_{i=1}^{n} E_{\underline{s}}\{c_i^2\} D(Q_i). \quad (8)$$

IV. Performance Analysis

In this section, we will analyze the performance characteristics of the LOR detector under the weakly dependent noise model. Let us define $\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c})\right\rangle_n = \sum_{i=1}^{n} \sum_{j=1}^{n} E_{\underline{s}}^2\{c_i c_j\}/n$, $\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle_n = \sum_{i=1}^{n} E_{\underline{s}}^2\{c_i^2\}/n$, $\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle_n = \sum_{i=1}^{n} E_{\underline{s}}\{c_i^2\}/n$, $I_1(f) = \int \left(\frac{f'(y)}{f(y)}\right)^2 f(y) dy$, and $I_2(f) = \int \left(\frac{f''(y)}{f(y)}\right)^2 f(y) dy$. We will drop the subscript n when n is infinite in the quantities defined with $\langle\cdot\rangle_n$. Then, the following theorem holds.

Theorem 1: The efficacy of the LOR detector for random signals is

$$\xi_{LOR} = 2I_1^2(f_e)\left[\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c})\right\rangle - \left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle\right] + I_2(f_e)\left[\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle - \left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle^2\right]. \quad (9)$$

Since the efficacy of the LO detector for random signals under the weakly dependent noise model is [12]

$$\xi_{LO} = 2I_1^2(f_e)\left[\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c})\right\rangle - \left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle\right] + I_2(f_e)\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle, \quad (10)$$

it is easily seen from (9) that the asymptotic performance of the LOR detector gets quite close to that of the LO detector when $\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle^2 \ll \left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle$. This condition means that the signal power is temporally quite nonhomogeneous and the correlation characteristics of the signal process are quite different from those of the noise process.

As an example, we consider the case where $r_s(i,j) = r^{|i-j|}$, $0 < |r| < 1$. Then we obtain the following quantities:

$$\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c})\right\rangle = \frac{(1-\rho r)K_1(\rho, r)}{(1+\rho r)^3(1-\rho^2)^3(1-r^2)}, \quad (11)$$

$$\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle = \frac{(1-\rho r)^2}{(1-\rho^2)^2(1+\rho r)^2}, \quad (12)$$

$$\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle = \frac{1-\rho r}{(1-\rho^2)(1+\rho r)}, \quad (13)$$

and

$$K_1(\rho, r) = (1+\rho^2 r^2)(1-r^2)(1-\rho^2) + 2(r-\rho)^2(1+\rho r)^2. \quad (14)$$

TABLE I

SOME QUANTITIES OF THE LO AND LOR DETECTORS WHEN $r_s(i,j) = r^{|i-j|}$.

$f_e(x)$	$e^{-\frac{x^2}{2}}/\sqrt{2\pi}$
$I_1(f_e)$	1
$I_2(f_e)$	2
ξ_{LO}	$\frac{2(1-\rho r)K_1(\rho,r)}{(1+\rho r)^3(1-\rho^2)^3(1-r^2)}$
ξ_{LOR}	$\frac{4(1-\rho r)K_2(\rho,r)}{(1+\rho r)^3(1-\rho^2)^3(1-r^2)}$
$ARE_{LOR,LO}$	$\frac{2K_2(\rho,r)}{K_1(\rho,r)}$
$K_1(\rho,r) = (1+\rho^2r^2)(1-r^2)(1-\rho^2) + 2(r-\rho)^2(1+\rho r)^2$	
$K_2(\rho,r) = \rho^2r^2(1-r^2)(1-\rho^2) + (r-\rho)^2(1+\rho r)^2$	

TABLE II

SOME QUANTITIES OF THE LO AND LOR DETECTORS WHEN $r_s(i,j) = r^{|i-j|}$.

$f_e(x)$	$\frac{e^{-x}}{(1+e^{-x})^2}$
$I_1(f_e)$	$\frac{1}{3}$
$I_2(f_e)$	$\frac{1}{5}$
ξ_{LO}	$\frac{(1-\rho r)K_3(\rho,r)}{45(1+\rho r)^3(1-\rho^2)^3(1-r^2)}$
ξ_{LOR}	$\frac{4(1-\rho r)K_2(\rho,r)}{9(1+\rho r)^3(1-\rho^2)^3(1-r^2)}$
$ARE_{LOR,LO}$	$\frac{20K_2(\rho,r)}{K_3(\rho,r)}$
$K_2(\rho,r) = \rho^2r^2(1-r^2)(1-\rho^2) + (r-\rho)^2(1+\rho r)^2$	
$K_3(\rho,r) = (9+11\rho^2r^2)(1-r^2)(1-\rho^2) + 20(r-\rho)^2(1+\rho r)^2$	

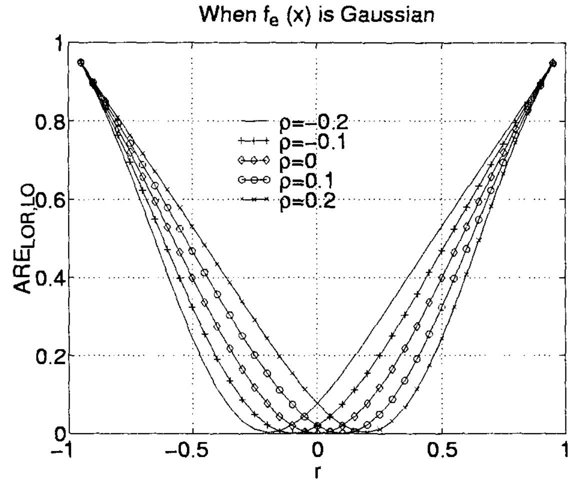

Fig. 1. $ARE_{LOR,LO}$ for various values of ρ when $f_e(x)$ is Gaussian.

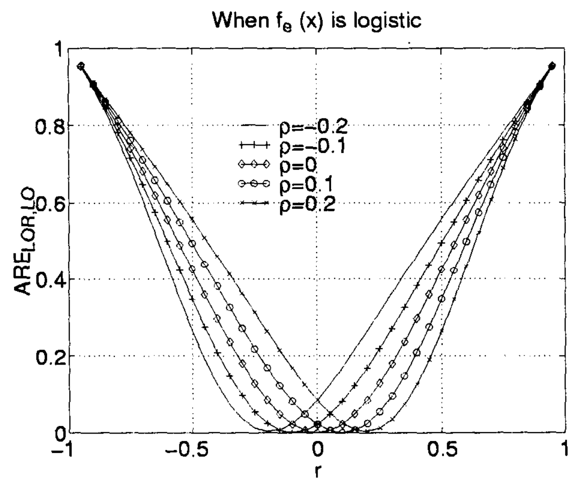

Fig. 2. $ARE_{LOR,LO}$ for various values of ρ when $f_e(x)$ is logistic.

The test statistic and efficacy of the LO detector are obtained in [12] and some asymptotic quantities of the LO and LOR detectors are shown in Tables I and II, when the additive noise is the first order MA of the i.i.d. Gaussian process and the i.i.d. logistic process. In addition, $ARE_{LOR,LO}$ curves are plotted for various values of ρ in Figs. 1 and 2. It is easily seen that the LOR detector is useful when the covariance function of the signal process is different from that of the noise process. That is, when $\rho = -0.2$ as an example, the value of the $ARE_{LOR,LO}$ is almost zero around $r = -0.2$ and gets larger as $|r - \rho|$ increases.

V. CONCLUDING REMARK

In this paper, we considered the LOR detection of random signals in a weakly dependent noise model. The test statistic of the LOR detector in weakly dependent noise was derived and the asymptotic performance of the LOR detector was analyzed and compared to that of the LO detector in terms of ARE.

The asymptotic performance of the LOR detector is worse than that of the LO detector as shown in Theorem 1. It was shown, however, that the LOR detector showed comparable performance to the LO detector in spite of its simple implementation when the signal power was temporally quite nonhomogeneous and the correlation characteristics of the signal process were different from those of the noise process.

APPENDIX

Proof of Theorem 1. It is easily seen that

$$\left.\frac{d^2E\{T_{LOR}(\underline{Y})|H_1\}}{d\theta^2}\right|_{\theta=0} = V\{T_{LOR}(\underline{Y})|H_0\}$$

$$= \quad E\{A^2 + 2AB + B^2|H_0\}, \tag{15}$$

where $A = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} E_{\underline{s}}\{c_i c_j\} z_i z_j \Gamma_2(Q_i, Q_j)$ and $B = \sum_{i=1}^{n} E_{\underline{s}}\{c_i^2\} D(Q_i)$. Let us define $< \Gamma_2^2 >_n = \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} \Gamma_2^2(i,j)/n(n-1)$ and $< D^m >_n = \sum_{i=1}^{n} D^m(i)/n$. Again, we will drop the subscript n when n is infinite. Then, it is easy to see that $< \Gamma_2^2 > = I_1^2(f_e)$, $< D^2 > = I_2(f_e)$, and $< D >_n = 0$ [7]. Thus, we obtain

$$\begin{aligned} E\{A^2\} &= 2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} E_{\underline{s}}^2\{c_i c_j\} E\{\Gamma_2^2(i,j)|H_0\} \\ &= 2n\left[\left\langle E_{\underline{s}}^2(\underline{c},\underline{c})\right\rangle_n - \left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle_n\right]\left\langle \Gamma_2^2\right\rangle_n, \end{aligned} \tag{16}$$

$$E\{AB\} \quad = \quad 0, \tag{17}$$

and

$$\begin{aligned} E\{B^2\} &= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} E_{\underline{s}}\{c_i^2\} E_{\underline{s}}\{c_j^2\} E\{D(Q_i)D(Q_j)|H_0\} \\ &\quad + \sum_{i=1}^{n} E_{\underline{s}}^2\{c_i^2\} E\{D^2(Q_i)|H_0\} \\ &= \frac{n^2}{n-1}\left[\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle_n - \left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle_n^2\right]\left\langle D^2\right\rangle_n . \end{aligned} \tag{18}$$

Therefore, we get

$$\begin{aligned} \xi_{LOR} &= \lim_{n\to\infty} \frac{V\{T_{LOR}(\underline{Y})\}}{n} \\ &= 2I_1^2(f_e)\left[\left\langle E_{\underline{s}}^2(\underline{c},\underline{c})\right\rangle - \left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle\right] \\ &\quad + I_2(f_e)\left[\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle - \left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle^2\right]. \end{aligned} \tag{19}$$

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