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Weak Random Signal Detection in a Weakly Dependent Noise Model

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Abstract

In this paper, we consider a discrete-time random signal detection problem under the presence of additive noise exhibiting weak dependence. We derive the test statistic of the locally optimum detector under a weakly dependent noise model. The performance characteristic of the locally optimum detector is analyzed and compared with that of the square-law detector in terms of asymptotic relative efficiency.

1 Introduction

The signal detection problem in noisy observations has been considered in many previous studies. Among the various signal detection problems, weak signal detection has been of much interest in detection theory and applications. Among the investigations on locally optimum detectors are those considered in [1]-[4].

It has been commonly assumed that the additive noise samples are statistically independent. In practice, however, the assumption is often violated, and the optimum detectors designed under the assumption are no longer optimum in practice. Thus, investigations on signal detection in dependent noise are required. Among the investigations on the detection problem under various dependent noise models are those in [5]-[7]. In these studies, however, detection schemes only for known signals were considered.

In this paper, we will investigate the locally optimum detection for random signals under a weakly dependent noise model. The weakly dependent noise will be modeled as the first order moving average of an i.i.d. random process.

2 Observation Model

In this paper, we will consider the detection problem of discrete-time random signals in weakly dependent noise environment. Let H_0 be the null hypothesis and H_1 be the alternative hypothesis. Then, the observation model can be written as

$$\begin{aligned} H_0: &\quad X_i = W_i, &\quad i = 1, 2, \cdots, n, \\ H_1: &\quad X_i = \theta s_i + W_i, &\quad i = 1, 2, \cdots, n, \end{aligned} \tag{1}$$

where $\{X_i\}$ are the observations, $\{W_i\}$ are the weakly dependent noise components, θ is a signal strength parameter, $\{s_i\}$ are the random signal components with mean zero and variance $\{\sigma_i^2\}$. Then, the detection problem becomes the problem of hypotheses decision based on the n

observations, $\{X_i\}$.

Weakly dependent noise can generally be modeled by the Volterra expansion [8]. This model, however, is almost intractable to handle because of the infinitely many terms of the expansion. In this paper, we will assume that the weakly dependent noise W_i, $i = 1, 2, \cdots, n$ are the moving average (MA) of i.i.d. random variables,

$$W_i = e_i + \rho e_{i-1} u_{i-2}, \tag{2}$$

where e_i, $i = 1, 2, \cdots, n$ are i.i.d. random variables with common pdf f_e, which is even symmetric with bounded continuous derivatives and satisfies the regularity condition [1]. Here, ρ is called the dependence parameter determining the correlation coefficient of W_i, and u_i is the unit step sequence.

Let $\underline{X}$, $\underline{w}$, $\underline{e}$ and $\underline{s}$ be the n-tuple vectors representing $(x_1, x_2, \cdots, x_n)$, $(W_1, W_2, \cdots, W_n)$, $(e_1, e_2, \cdots, e_n)$, and $(s_1, s_2, \cdots, s_n)$, respectively, and $f_{\underline{w}}(\underline{w})$, $f_{\underline{e}}(\underline{e}) = \prod_{i=1}^n f_e(e_i)$, and $f_{\underline{s}}(\underline{s})$ be the pdfs of $\underline{w}$, $\underline{e}$, and $\underline{s}$, respectively. Then, we have

$$\begin{aligned} f_{\underline{w}}(\underline{w}) &= f_e(W_1) f_e(W_2 - \rho W_1) \cdots \\ &\quad f_e(W_n - \rho W_{n-1} + \cdots + (-\rho)^{n-1} W_1) \\ &= f_e(X_1 - \theta s_1) f_e(X_2 - \theta s_2 - \rho(X_1 - \theta s_1)) \cdots \\ &\quad f_e(X_n - \rho X_{n-1} + (-\rho)^{n-1} X_1 \\ &\quad -\theta(s_n - \rho s_{n-1} + \cdots + (-\rho)^{n-1} s_1) \\ &= \prod_{i=1}^n f_e(Y_i - \theta c_i) \\ &= f_{\underline{e}}(\underline{Y} - \theta \underline{c}), \end{aligned} \tag{3}$$

where $Y_i = \sum_{k=0}^{i-1} (-\rho)^k X_{i-k}$, $c_i = \sum_{k=0}^{i-1} (-\rho)^k s_{i-k}$, $\underline{Y} = (Y_1, Y_2, \cdots, Y_n)$, and $\underline{c} = (c_1, c_2, \cdots c_n)$.

3 The Locally Optimum Detector

Let us define

$$\phi(\underline{X}|\theta) = \int_{R^n} f_{\underline{w}}(\underline{X} - \theta \underline{s}) f_{\underline{s}}(\underline{s}) d\underline{s}, \tag{4}$$

where R^n is the set of all n-tuples of real numbers. Then, the locally optimum (LO) test statistic is [1]

$$T_{LO}(\underline{X}) = \frac{\left.\frac{d^\nu \phi(\underline{X}|\theta)}{d\theta^\nu}\right|_{\theta=0}}{\phi(\underline{X}|0)}, \tag{5}$$

where ν is the order of the first nonzero derivative of $\phi(\underline{X}|\theta)$ at $\theta = 0$.

From (3) and (4), it is easily seen that

$$\begin{aligned} \left.\frac{d^2 \phi(\underline{X}|\theta)}{d\theta^2}\right|_{\theta=0} &= \int_{R^n} \left.\frac{d^2 f_{\underline{e}}(\underline{Y} - \theta \underline{c})}{d\theta^2}\right|_{\theta=0} f_{\underline{s}}(\underline{s}) d\underline{s} \\ &= \int_{R^n} f_{\underline{e}}(\underline{Y}) f_{\underline{s}}(\underline{s}) \left[\sum_{i=1}^n \sum_{j=1, j\neq i}^n c_i c_j \cdot \right. \\ &\quad \left. g_{LO}(Y_i) g_{LO}(Y_j) \sum_{i=1}^n c_i^2 h_{LO}(Y_i) \right] d\underline{s} \end{aligned} \tag{6}$$

and

$$\begin{aligned} \phi(\underline{X}|0) &= \int_{R^n} f_{\underline{e}}(\underline{Y}) f_{\underline{s}}(\underline{s}) d\underline{s} \\ &= f_{\underline{e}}(\underline{Y}), \end{aligned} \tag{7}$$

where $g_{LO}(x) = -f_e'(x)/f_e(x)$ and $h_{LO}(x) = f_e''(x)/f_e(x)$. Then, the LO test statistic can be obtained as

$$\begin{aligned} T_{LO}(\underline{Y}) &= \sum_{i=1}^n \sum_{j=1, j\neq i}^n E_{\underline{s}}\{c_i c_j\} g_{LO}(Y_i) g_{LO}(Y_j) \\ &\quad + \sum_{i=1}^n E_{\underline{s}}\{c_i^2\} h_{LO}(Y_i), \end{aligned} \tag{8}$$

where $E_{\underline{s}}\{\cdot\}$ is the expectation over $\underline{s}$.

4 Performance Analysis

In this section, we will analyze the performance characteristics of the LO detector under the weakly dependent noise model. The performance of the LO detector will be compared with that of the square-law (SQ) detector whose test statistic is

$$T_{SQ} = \sum_{i=1}^n X_i^2. \tag{9}$$

Theorem 1 *The efficacy of the LO detector is*

$$\begin{aligned} \xi_{LO} &= 2 I_1^2(f_e) \left[\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c}) \right\rangle - \left\langle E_{\underline{s}}^2(\underline{c}^2) \right\rangle \right] \\ &\quad + I_2(f_e) \left\langle E_{\underline{s}}^2(\underline{c}^2) \right\rangle, \end{aligned} \tag{10}$$

where

$$\left\langle E_{\underline{s}}^2(\underline{c}, \underline{c}) \right\rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n E_{\underline{s}}^2\{c_i, c_j\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}Q_{2,i,k}(-\rho)\cdot Q_{2,j,l}(-\rho)r_s(i,j)r_s(k,l), \quad (11)$$

$$\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E_{\underline{s}}^2\{c_i^2\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}Q_{4,i,j,k,l}(-\rho)\cdot r_s(i,j)r_s(k,l), \quad (12)$$

$$I_1(f) = \int\left(\frac{f'(y)}{f(y)}\right)^2 f(y)dy, \quad (13)$$

$$I_2(f) = \int\left(\frac{f''(y)}{f(y)}\right)^2 f(y)dy, \quad (14)$$

$$Q_{2,i,j}(x) = (x^{2\max(i,j)-i-j} - x^{2n-i-j+2})/(1-x^2), \quad (15)$$

$$Q_{4,i,j,k,l}(x) = (x^{4\max(i,j,k,l)-i-j-k-l} - x^{4n-i-j-k-l+4})/(1-x^4), \quad (16)$$

and

$$r_s(i,j) = E_{\underline{s}}\{s_i s_j\}. \quad (17)$$

Theorem 2 *The efficacy of the SQ detector is*

$$\xi_{SQ} = \frac{4\left((1-\rho^2)\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle + 2\rho\left\langle E_{\underline{s}}(\underline{s},\underline{c})\right\rangle\right)^2}{(1+\rho^2)^2 m_4 - (1-\rho^2)^2\sigma_e^4}, \quad (18)$$

where

$$\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E_{\underline{s}}\{c_i^2\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}Q_{2,i,j}(-\rho)r_s(i,j), \quad (19)$$

$$\left\langle E_{\underline{s}}(\underline{s},\underline{c})\right\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}E_{\underline{s}}\{s_i c_{i-1}\}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{i-1}(-\rho)^{i-j-1}r_s(i,j), \quad (20)$$

$$m_4 = \int x^4 f_e(x)dx, \quad (21)$$

and

$$\sigma_e^4 = \left(\int x^2 f_e(x)dx\right)^2. \quad (22)$$

The proofs of Theorem 1 and 2 are shown in [9].

Now, let us consider some examples to show the asymptotic performance of the locally optimum detector more explicitly.

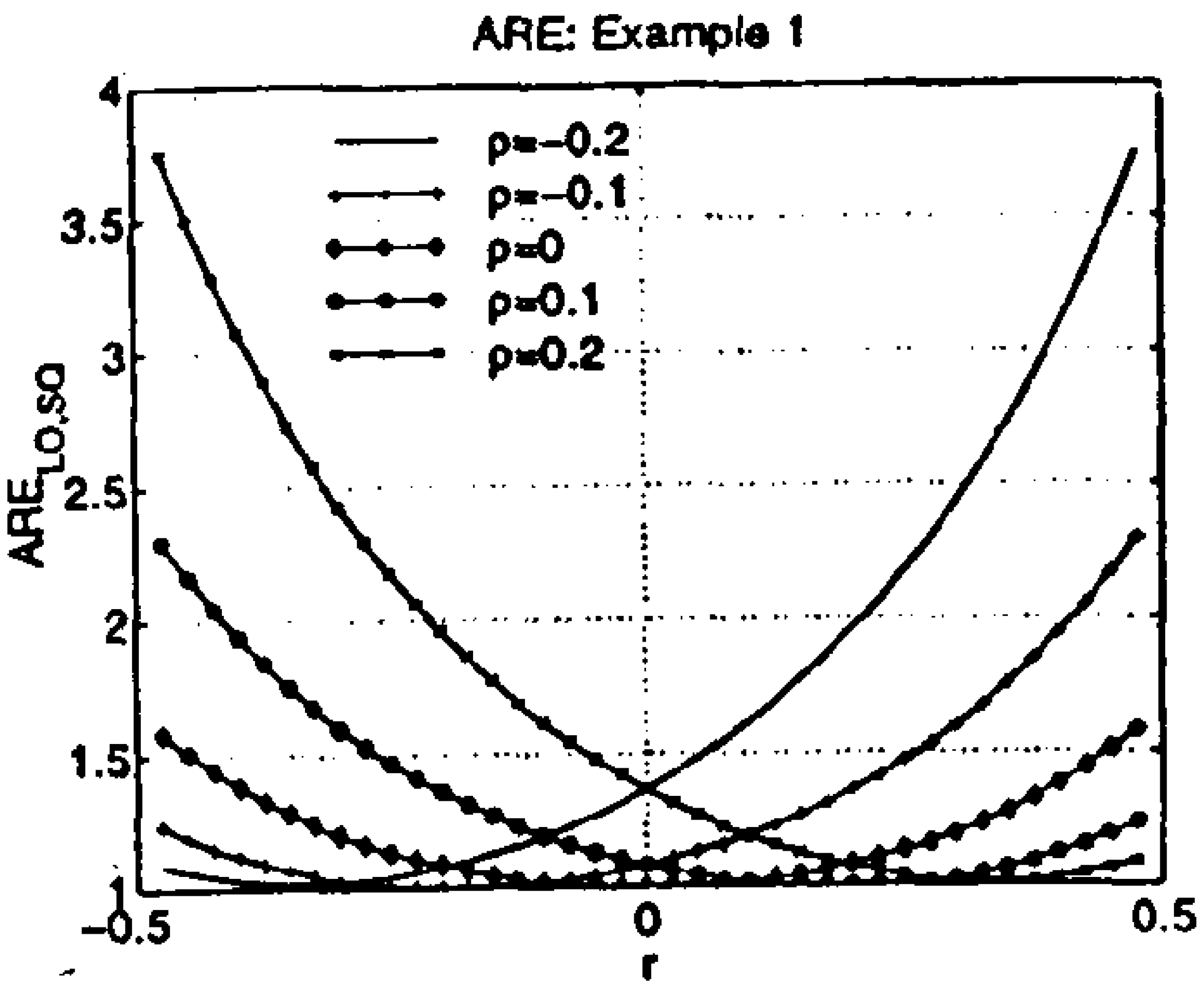

Figure 1: $ARE_{LO,SQ}$ for various values of ρ when the noise is the first order MA of an i.i.d. Gaussian process.

Example 1. Let the covariance function be $r_s(i,j) = r^{|i-j|}$, where $0 < |r| < 1$ and $f_e(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$. Then, we can obtain $I_1(f_e) = 1$, $I_2(f_e) = 2$, $m_4 = 3$, $\sigma_e^4 = 1$,

$$\left\langle E_{\underline{s}}^2(\underline{c},\underline{c})\right\rangle = \frac{(1-\rho r)K(\rho,r)}{(1+\rho r)^3(1-\rho^2)^3(1-r^2)}, \quad (23)$$

$$\left\langle E_{\underline{s}}^2(\underline{c}^2)\right\rangle = \frac{(1-\rho r)^2}{(1-\rho^2)^2(1+\rho r)^2}, \quad (24)$$

$$\left\langle E_{\underline{s}}(\underline{c}^2)\right\rangle = \frac{1-\rho r}{(1-\rho^2)(1+\rho r)}, \quad (25)$$

$$\left\langle E_{\underline{s}}(\underline{s},\underline{c})\right\rangle = \frac{r}{1+\rho r}, \quad (26)$$

and

$$K(\rho,r) = (1+\rho^2r^2)(1-r^2)(1-\rho^2) + 2(r-\rho)^2(1+\rho r)^2. \quad (27)$$

Then, from Theorems 1 and 2, the $ARE_{LO,SQ}$ is

$$ARE_{LO,SQ} = \frac{(1-\rho r)(1+4\rho^2+\rho^4)K(\rho,r)}{(1-\rho^2)^3(1+\rho r)^3(1-r^2)}. \quad (28)$$

Example 2. Let the covariance function be $r_s(i,j) = r^{|i-j|}$, where $0 < |r| < 1$ and $f_e(x) = \frac{e^{-x}}{(1+e^{-x})^2}$. Then, we can obtain $I_1(f_e) = \frac{1}{3}$, $I_2(f_e) = \frac{1}{5}$, $m_4 = \frac{7\pi^4}{15}$, and $\sigma_e^4 = \frac{\pi^4}{9}$. Then, from Theorems 1 and 2, the $ARE_{LO,SQ}$ is

$$ARE_{LO,SQ} = \frac{\pi^4(4+13\rho^2+4\rho^4)}{2025}\cdot\left(\frac{10(1-\rho r)K(\rho,r)}{(1+\rho r)^3(1-\rho^2)^3(1-r^2)} - \frac{(1-\rho r)^2}{(1-\rho^2)^2(1+\rho r)^2}\right). \quad (29)$$

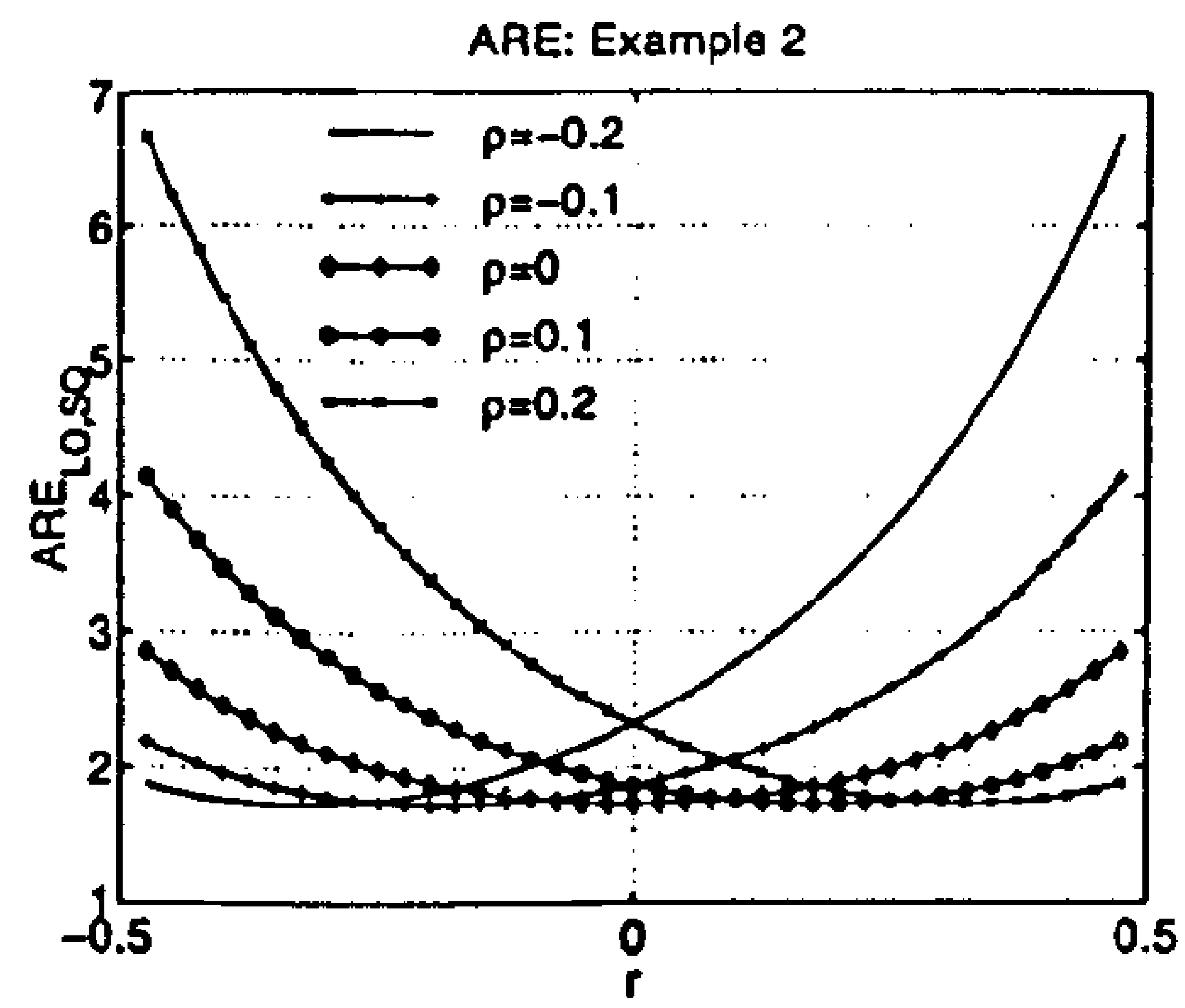

Figure 2: $ARE_{LO,SQ}$ for various values of ρ when the noise is the first order MA of the i.i.d. symmetric logistic process.

In Figs. 1 and 2, the $ARE_{LO,SQ}$ derived in the two examples are plotted for various values of ρ when the additive noise is the first order MA of an i.i.d. Gaussian process and an i.i.d. symmetric logistic process, respectively.

5 Concluding Remark

In this paper, we considered the locally optimum detection of random signals in additive weakly dependent noise. The test statistic of the locally optimum detector for random signals in weakly dependent noise was derived and the asymptotic performance of the locally optimum detector was analyzed and compared to that of the square-law detector.

Acknowledgement

This research was supported by Korea Science and Engineering Foundation (KOSEF) under Grant 971-0916-097-2, for which the authors would like to express their thanks.

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